Basic solution (B.S)

Consider a system of m linear simultaneous equations with n variables: $Ax = b$ $(m < n)$

let B = any $m \times m$ submatrix formed by m linearly independent column of A.

Then the solution obtained by setting $(n-m)$ variables not associated with the columns of B, equal to zero and solving the resulting system of equations, is called a basic solution to the given system provided $\det B \neq 0$.

The m variables which may be different from zero, are known as basic variables and the remaining $(n-m)$ variables are called the non-basic variables.

<u>Note</u>: For the chosen submatrix B, the basic solution to the system can be obtained as

$$B x_B = b$$
$$\Rightarrow x_B = B^{-1} b$$

x_B = basic solution

~~Basi~~.

Feasible Solution (F.S.)

A solution of a L.P.P. satisfying all the constraints and non-negativity conditions is called a feasible solution.

Basic feasible solution (B.F.S.)

A feasible solution to a L.P.P, which is also basic solution to the problem, is known as its basic feasible solution.

Degenerate and Non-degenerate B.F.S :

A basic feasible solution is said to be degenerate B.F.S if it contains at least one zero basic variable. Otherwise, the B.F.S. is called non-degenerate B.F.S.

Note: In algebraic solution space defined by ($m \times n$ equations, $m < n$), basic solutions correspond to the corner points in the graphical solution space.

The maximum number of corner points (Hence no. of Basic solutions)

$$= {}^{n}C_{m}$$

$$= \frac{n!}{m!\,(n-m)!}$$

Remark 1:
If a L.P.P. has a basic solution, then it has also a basic feasible solution.

Remark 2: The number of basic solutions are at most ${}^{n}C_{m}$. Therefore the maximum number of basic F.S. to an L.P.P. is ${}^{n}C_{m}$ which is finite.

Remark 3: If the given L.P.P. has an optimal solution then at least one B.F.S is optimal.

Q Consider the system $x_1 + x_1 + x_3 = 8$
$2x_1 + x_2 + x_4 = 10$
$x_1, x_2, x_3, x_4 \geq 0$.

Find all basic solutions. Also identify those basic solutions which are B.F.S. Are there any degenerate B.F.S?

Ans: Here $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $b = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$

maximum no. of basic solutions $= {}^nC_m = {}^4C_2 = \frac{4!}{2!\,2!}$
$= \frac{4 \times 3}{2} = 6$

Now,
(i) $|B_1| = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \neq 0$ ∴ B_1 = basis matrix
x_1, x_2 = basic variables
x_3, x_4 = non " "

setting $x_3 = x_4 = 0$,
$x_1 + x_2 = 8$
$2x_1 + x_2 = 10$ $\Rightarrow x_1 = 2, \quad x_2 = 6$

∴ Basic solution is $x_1 = 2, x_2 = 6, x_3 = 0, x_4 = 0$

Further, as both basic variables are strictly positive this is a non-degenerate B.F.S, corresponds to B_1.

(ii) $|B_2| = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2 \neq 0$

Non-degenerate B.F.S.: $x_1 = 5, x_2 = 0, x_3 = 3, x_4 = 0$

(iii) $|B_3| = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 \neq 0$

Basic solution is: $x_1 = 8, x_2 = 0, x_3 = 9$ $x_4 = -6$
which is not a B.F.S. as x_4 is negative.

(iv) $|B_4| = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$, Basic solution: $x_1 = 0, x_4 = 0, x_2 = 10$
$x_3 = -2$
which is not a B.F.S. as x_3 is negative

(v) $|B_5| = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \neq 0$, Non-degenerate B.F.S: $x_1 = 0, x_2 = 8, x_3 = 0$ $x_4 = 2$

(vi) $|B_6| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0$, " " : $x_1 = 0, x_2 = 0, x_3 = 8, x_4 = 10$

Remark 1:

Every corner points of the feasible region is a B.F.S. of the system $AX = b$ $X \geq 0$; and conversely every B.F.S. of the above system is a corner point of the set (feasible solution set).

Further for the non-degenerate B.F.S, the correspondence is one to one, i.e. two distinct non-degenerate B.F. Solutions will correspond to two distinct corner points of the feasible set, but for the degenerate basic feasible solutions, more than one degenerate B.F. Solutions may correspond to the same corner point.

Associated cost vector:

Let the given LPP is optimize $Z = CX$

s.t $AX = b$, $X \geq 0$

Let X_B = be a B.F.S to the above L.P.P.

Then the vector $C_B = (C_{B1}, C_{B2}, \ldots, C_{Bm})$, where C_{Bi} $(i=1,2,\ldots,n)$ are the components of C_B associated with the basic variables, is known as the cost vector associated with the B.F.S X_B.

Then the value of the objective function corresponding to the B.F.S X_B is

$$Z_B = C_B X_B$$

Algorithm of the simp[illegible]

Step 1: Convert the given LPP into standard form.

Step 2: Construct the initial simplex table and find initial B.F.S.

Step 3: Calculation of Z and Δ_j (Net evaluation) and test the basic feasible solution for optimality by the rules given below:

$$Z = C_B x_B \quad \text{and} \quad \Delta_j = z_j - C_j = C_B x_j - C_j$$

where

$x_B = [x_{B1}, x_{B2}, \ldots, x_{Bm}]^T$, x_{Bi} = Basic variables

$C_B = [C_{B1}, C_{B2}, \ldots, C_{Bm}]^T$, C_{Bi} = coefficients of basic variables $(i = 1, 2, 3, \ldots, m)$

Rule 1: If all $Z_j - C_j \geq 0$, the current B.F.S. is optimal.

Rule 2: If some $Z_j - C_j < 0$ and for that some elements of the column x_j are positive (i.e. $y_{ij} > 0$), then there exists a new B.F.S called improved B.F.S. proceed to improve the solution in the next step.

Rule 3: If some $Z_j - C_j < 0$ and corresponding to that all elements of the column x_j are negative or zero (i.e. $y_{ij} \leq 0$) then the given LPP has unbounded solution.

Step 4: To improve the B.F.S., the incoming and outgoing vectors are determined

i) Incoming (Entering) vector: most negative value of $Z_j - C_j$

ii) Outgoing (Leaving) vector: $\min \left\{ x_B / x_i, \; x_i > 0 \right\}$

Step 5: Mark the key element (pivot element) at the intersection of incoming vector and outgoing vector.

Step 6. Now repeat step 3 to step 5 untill an optimal solution is obtained.

Q(1) Solve the following LPP by Simplex method

$$\text{Max } Z = 80x_1 + 55x_2$$

subject to $4x_1 + 2x_2 \le 40$

$2x_1 + 4x_2 \le 32$

$x_1, x_2 \ge 0$

$Z = C_B X_B$

$\Delta_j = Z_j - C_j$

$= C_B X_{B_j} - C_j$

most negative of $Z_j - C_j$

Ans: Introducing slack variables S_1 and S_2, the given LPP can be put in the standard form as:

$$\text{Max } Z = 80x_1 + 55x_2 + 0.S_1 + 0.S_2$$

s.t $4x_1 + 2x_2 + S_1 = 40$

$2x_1 + 4x_2 + S_2 = 32$

$x_1, x_2, S_1, S_2 \ge 0.$

↑ = incoming vector

← = outgoing vector

		C_j	80	55	0	0	Min Ratio	
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	X_B/x_1	
← S_1	0	40	4	2	1	0	$\frac{40}{4}=10$	
S_2	0	32	2	4	0	1	$\frac{32}{2}=16$	
$Z = C_B X_B = 0$		$\Delta_j = Z_j - C_j =$	-80 ↑	-55	0	0	$\min(X_B/x_2)$	
x_1	80	10	1	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{10}{1/2}=20$	$R_2' = R_2 - 2R_1'$
← S_2	0	12	0	3	$-\frac{1}{2}$	1	$\frac{12}{3}=4$	
$Z = 800$		$Z_j - C_j =$	0	-15 ↑	20	0		
x_1	80	8	1	0	$1/3$	$-1/6$		$R_1' = R_1 - \frac{1}{2}R_2'$
x_2	55	4	0	1	$-1/6$	$1/3$		
$Z = 860$		$Z_j - C_j$	0	0	$35/2$	5		

Since all $Z_j - C_j \ge 0$, optimal basic feasible solution is obtained.

∴ $\text{Max } Z = 860, \quad x_1 = 8, \quad x_2 = 4$